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A NOTE ON GEOMETRY OF POLYHEDRON OF THREE-INDEX TRANSPORTATION PROBLEM

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Abstract. We show that the polyhedron of three-index transportation problem is the intersection of the positive orthant and three mutually orthogonal affine sets.

Key words and phrases: three-index transportation problem, linear algebra

1. INTRODUCTION

Recently it was shown in [1] that the polihedron of transportation problem is the intersection of the positive orthant and two mutually orthogonal affine sets. In this paper we show an analogon of this result for three-index transportation problem, planar as well as axial. This can be of interest in applying algorithms which use projections on this polihedron.

The polyhedron of the planar three-index transportation problem is the set of nonnegative solutions of the following system:

$$\sum_{j=1}^{\beta}\sum_{k=1}^{\gamma} x_{ijk} = a_i, \qquad i = 1, 2, \ldots, \alpha$$

$$\sum_{i=1}^{\alpha}\sum_{k=1}^{\gamma} x_{ijk} = b_j, \qquad j = 1, 2, \ldots, \beta$$

$$\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} x_{ijk} = c_k, \qquad k = 1, 2, \ldots, \gamma.$$

Denote by $a = (a_i)$, $b = (b_j)$, $c = (c_k)$, $x = (x_{ijk})$. We can rewrite the system as

$$Ax = a \tag{1}$$

$$Bx = b \tag{2}$$

$$Cx = c \tag{3}$$

where A, B, C are appropriate matrices. Assume that the system has a solution. We say that affine sets given by (1) and (2) are orthogonal, iff linear spaces $\{x \mid Ax = 0\}$ and $\{x \mid Bx = 0\}$ can be represented respectively in the form $V \oplus V_1$ and $V \oplus V_2$, for some linear spaces V, V_1, V_2, such that V_1 and V_2 are nontrivial and $v_1^T v_2 = 0$, for every $v_1 \in V_1$ and $v_2 \in V_2$.

2. MAIN RESULTS

Our main result can be stated as follows.

THEOREM 1. *Affine sets given by* (1), (2) *and* (3) *are mutually orthogonal.*

PROOF. We show first that affine sets given by (1) and (2) are orthogonal. Orthogonality of other pairs follows by simmetry. Let P, Q be projection matrices respectively onto affine sets (1) and (2). The orthogonality of these sets will be shown if we prove that $P \cdot Q = Q \cdot P$. Denote by $M_{k,l,m}$ the $l \times km$ matrix consisting of $1 \times m$ blocks of the type

$$\begin{bmatrix} 1 & \dots & 1 & & & & & & \\ & & & 1 & \dots & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \dots & 1 \end{bmatrix}$$

having l rows with k units in each of them (nonwritten elements are zero). Remark that $A = M_{\beta\gamma,\alpha,1}$, $B = M_{\gamma,\beta,\alpha}$. The projection matrix onto $\{x \mid Mx = 0\}$, where $M = M_{k,l,m}$ is

$$I - M^T(MM^T)^{-1}M$$

(I is the appropriate unit matrix). Let I_k be the unit matrix of the format $k \times k$ and E_k a matrix of the same size with unit elements. Than $MM^T = km\ I_l$, and

$$M^T(MM^T)^{-1}M = \frac{1}{km}N,$$

where N consists of $m \times m$ blocks of the type $\operatorname{diag}(\overbrace{E_k, \dots, E_k}^{l})$. If we denote by $X = \operatorname{diag}(\overbrace{E_{\beta\gamma}, \dots, E_{\beta\gamma}}^{\alpha})$ and by Y the matrix of $\alpha \times \alpha$ blocks of the type $\operatorname{diag}(\overbrace{E_\gamma, \dots, E_\gamma}^{\beta})$, we find that

$$P = I_{\alpha\beta\gamma} - \frac{1}{\beta\gamma}X \qquad \text{and}$$

$$Q = I_{\alpha\beta\gamma} - \frac{1}{\alpha\gamma}Y.$$

Using rules for multiplication of blocks matrices we find

$$X \cdot Y = Y \cdot X = \gamma E_{\alpha\beta\gamma}.$$

Thus $P \cdot Q = Q \cdot P$ and the proof is completed.

We state now the similar result for axial three-index transportation problem. Recall that in this case the polyhedron is the set of nonnegative solutions of the system

$$\sum_{i=1}^{\alpha} x_{ijk} = a_{jk}, \qquad j = 1, \dots, \beta; \ k = 1, \dots, \gamma \tag{4}$$

$$\sum_{j=1}^{\beta} x_{ijk} = b_{ik}, \qquad i = 1, \dots, \alpha; \ k = 1, \dots, \gamma \tag{5}$$

$$\sum_{k=1}^{\gamma} x_{ijk} = c_{ij}, \qquad i = 1, \dots, \alpha; \ j = 1, \dots, \beta \tag{6}$$

Assume that this system is possible.

THEOREM 2. *The affine sets given by* (4), (5) *and* (6) *are mutually orthogonal.*

Since the proof uses the same method as in Theorem 1, we omit it.

3. CONCLUSION

Accoding to theorems 1 and 2, we can find projection of a vector onto affine sets given by (1)–(3) (or (4)–(6)) as composition of projections onto separate affine sets (1), (2), (3) (or (4), (5), (6)) in arbitrary order. An algorithm for transportation problem which uses this nice property is under investigation.

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